

So, when we know with some accuracy the position of q at time 0,

then we cannot specify its momentum at any time t .

Lectures 29 + 30:

①

Gaussians:

We will consider the centered

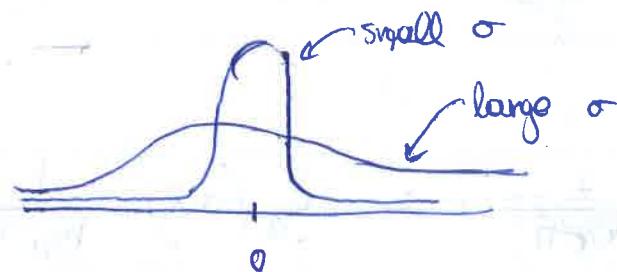
Gaussians

$$g_0(x) = \frac{1}{\sqrt{2\pi\sigma^2}} \cdot e^{-\frac{x^2}{2\sigma^2}}$$

This is the probability density of the normal distribution with mean 0 and standard deviation σ .

Notice that this knowledge immediately reveals

that $\int_{-\infty}^{+\infty} \frac{1}{\sqrt{2\sigma^2\pi}} e^{-\frac{x^2}{2\sigma^2}} dx = 1$



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We will show that, for the simple Gaussian

$$g_1(x) = \frac{1}{\sqrt{2\pi}} \cdot e^{-\frac{x^2}{2}},$$

its Fourier transform is practically itself:

$$\hat{g}_1(x) = \frac{1}{\sqrt{2\pi}} e^{-\frac{x^2}{2}}, \quad \forall x \in \mathbb{R}.$$

As the rest of the Gaussians are essentially dilations of g_1 , so are their Fourier transforms:

$$g_\sigma(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2}{2\sigma^2}} = \frac{1}{\sqrt{2\sigma^2\pi}} \cdot e^{-\frac{(\frac{x}{\sigma})^2}{2}} = \\ = \frac{1}{\sigma} \cdot g_1\left(\frac{1}{\sigma} \cdot x\right), \quad \forall x \in \mathbb{R}.$$

Thus, $\hat{g}_\sigma(x) = \frac{1}{\sigma} \cdot \hat{g}_1\left(\frac{1}{\sigma} \cdot (\cdot)\right)(x) =$

$$= \frac{1}{\sigma} \cdot \frac{1}{\sqrt{2\pi}} \cdot \hat{g}_1\left(\frac{x}{\sigma}\right) =$$

$$= \frac{1}{2n} e^{-\left(\frac{x}{\frac{1}{\sigma}}\right)^2} = \frac{\frac{1}{\sqrt{2n\sigma^2}}}{\frac{\sqrt{2n}(\frac{1}{\sigma})^2}{2n}} g_{1/\sigma}(x), \forall x \in \mathbb{R}. \quad (3)$$

That is, the Gaussian g_0 , which is mainly localised on $[-\sigma, \sigma]$, has Fourier transform

$\frac{1}{\sqrt{2n\sigma^2}} g_{1/\sigma}$, the multiple of a Gaussian;

that is mainly localised on $[-\frac{1}{\sigma}, \frac{1}{\sigma}]$.

This is a perfect demonstration of the uncertainty principle.

To recap:

→ $\hat{g}_0(x) = \frac{1}{2n} e^{-\frac{x^2}{\sigma^2}}, \forall x \in \mathbb{R}.$

This follows from $\hat{g}_1(x) = \frac{1}{2n} e^{-\frac{x^2}{\sigma^2}}, \forall x \in \mathbb{R}.$

↪ Proof: (if you are interested). $\hat{g}_1(x) = \frac{1}{\sqrt{2n}} e^{-\frac{x^2}{\sigma^2}}, \forall x \in \mathbb{R}.$

So, $\hat{g}_1'(x) = \frac{1}{\sqrt{2n}} \cdot \left(-\frac{x^2}{\sigma^2}\right)' \cdot e^{-\frac{x^2}{\sigma^2}} = -x \cdot \hat{g}_1(x)$, i.e.

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$$g_1'(x) = -x g_1(x), \quad \forall x \in \mathbb{R}. \quad \textcircled{*}$$

We apply the Fourier transform on both sides of $\textcircled{*}$:

- The LHS is easy: we know that $\widehat{g_1'}(x) = (ix) \cdot \widehat{g_1}(x), \quad \forall x \in \mathbb{R}.$

- For the RHS: $\widehat{-xg_1(x)} = -i(\widehat{g_1})'$:

$$-i(\widehat{g_1})'(y) = -i \frac{d}{dy} \left(\frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{-iyx} g_1(x) dx \right) =$$

$$= -i \cdot \frac{1}{2\pi} \cdot \int_{-\infty}^{+\infty} \left(\frac{d}{dy} e^{-iyx} \right) g_1(x) dx =$$

$$= -i \cdot \frac{1}{2\pi} \cdot \int_{-\infty}^{+\infty} (-ix) \cdot e^{-iyx} g_1(x) dx =$$

$$= (-x \widehat{g_1(x)})' (y)$$

Thus, $g_1'(x) = -x g_1(x) \quad \forall x \in \mathbb{R} \rightarrow (ix) \widehat{g_1}(x) = -i(\widehat{g_1})'(x), \quad \forall x \in \mathbb{R}$

$$\Leftrightarrow \frac{(\widehat{g_1})'(x)}{\widehat{g_1}(x)} = -x \quad \forall x \in \mathbb{R} \Rightarrow$$

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$$\rightarrow \int_0^x \frac{(\hat{g}_1)'(y)}{\hat{g}_1(y)} dy = - \int_0^x y dy$$

" $(\ln \hat{g}_1(y))'$

$$\rightarrow \ln \hat{g}_1(x) - \underbrace{\ln \hat{g}_1(0)}_{\substack{|| \\ 1}} = -\frac{x^2}{2} \quad \forall x \in \mathbb{R}$$

$$\frac{1}{2n} \underbrace{\int_{-\infty}^{+\infty} g_1(x) dx}_{\substack{|| \\ 1}} = \frac{1}{2n}$$

$$\Leftrightarrow \ln \hat{g}_1(x) + \ln(2n) = -\frac{x^2}{2} \quad \forall x \in \mathbb{R}$$

$$\Leftrightarrow \ln(2n \cdot \hat{g}_1(x)) = -\frac{x^2}{2} \quad \forall x \in \mathbb{R}$$

$$\Leftrightarrow \ln \hat{g}_1(x) = e^{-\frac{x^2}{2}} \quad \forall x \in \mathbb{R}$$

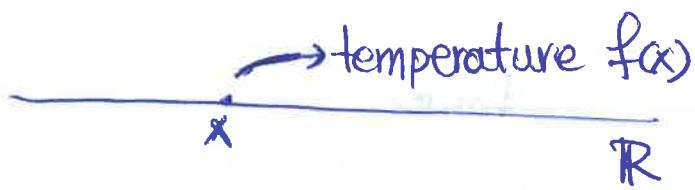
$$\Leftrightarrow \hat{g}_1(x) = \frac{1}{2n} \cdot e^{-\frac{x^2}{2}}, \quad \forall x \in \mathbb{R}$$

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Gaussians appear a lot in nature. For instance, they are central when it comes to diffusion of heat. We will solve now the heat equation on \mathbb{R} using the Fourier transform, and then we will introduce convolutions, to understand the solution better.

Heat equation, part I:

Let $f: \mathbb{R} \rightarrow \mathbb{R}$ be the temperature distribution of an infinitely long "string" at time 0;



i.e., suppose that the temperature of the point x is $f(x)$ at time 0.

What is the temperature $u(x,t)$ at time t ?

u is given by the heat equation :

constant,
depends only on the material

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$$*\Delta_x u(x,t) = \frac{\partial}{\partial t} u(x,t)$$

$$u(x,0) = f(x), \quad \forall x \in \mathbb{R}, \quad \text{for } f \in L^1(\mathbb{R}).$$

We take the Fourier transform w.r.t. on both sides:

$$\Rightarrow \underbrace{a \Delta_x u(x,t)}_{||} = \underbrace{\frac{\partial}{\partial t} u(x,t)}_{||}$$

$$a \cdot (ix)^2 \hat{u}(x,t) \quad \frac{\partial}{\partial t} \hat{u}(x,t)$$

$$\Rightarrow -ax^2 \hat{u}(x,t) = \frac{\partial}{\partial t} \hat{u}(x,t)$$

$$\Rightarrow \hat{u}(x,t) = e^{-ax^2 t} c(x),$$

where $\hat{u}(x,0) = e^{-ax^2 \cdot 0} c(x) = c(x)$,

i.e. $c(x) = \hat{u}(x,0) = \hat{f}(x)$

Thus,

$$\boxed{\hat{u}(x,t) = e^{-ax^2 t} \cdot \hat{f}(x)}, \quad \forall x \in \mathbb{R}, \forall t \in \mathbb{R}.$$

Observe that, since we know the Fourier transform of $u(\cdot, t) \forall t \in \mathbb{R}$, we can get $u(\cdot, t) \forall t \in \mathbb{R}$, by the Fourier inversion formula.

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However, there are things Fourier inversion doesn't reveal at first glance.

For instance, we expect that heat should diffuse, and should also be real, and that is not all that clear from

the fact that $u(x, t) = \int_{-\infty}^{+\infty} e^{ixy} \cdot e^{-ay^2} f(y) dy \dots$

$$-\frac{x^2}{2 \cdot \left(\frac{1}{\sqrt{2at}}\right)^2}$$

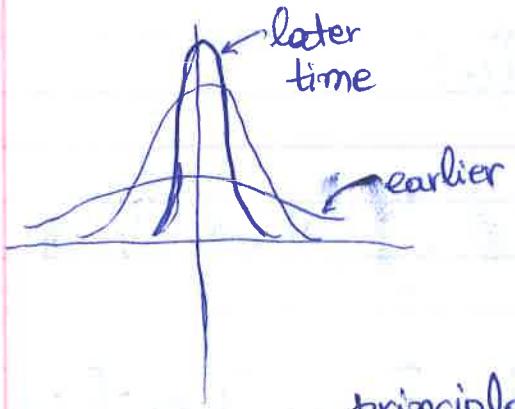
But: $\hat{u}(x, t) = e^{-ax^2 t} \hat{f}(x)$ $\forall x \in \mathbb{R}$, for all times t ,

where $e^{-ax^2 t}$ is a (multiple of a) Gaussian,

with standard deviation $\frac{1}{\sqrt{2at}}$, which decreases

as time passes. So, $e^{-ax^2 t}$ becomes a more

and more concentrated Gaussian as time passes:



So, $e^{-ax^2 t} \cdot \hat{f}(x)$ gets more

and more localised as time

passes, so, by the uncertainty

\rightarrow the temperature

$u(x, t)$ should get more and

more spread out. We see that $\hat{u}(x, t)$ gives

more information than the formula for u above.

We want a formula for u that will reveal all this...

This is where convolutions come in.

Notice that the solution $\hat{u}(x,t)$ equals

$$\underbrace{e^{-\alpha x t}}_{\text{the Fourier transform of another Gaussian}} \cdot \underbrace{\hat{f}(x)}_{\text{a Fourier transform}} = \mathcal{L}^{-1} \cdot \hat{g}_\sigma(x) \cdot \hat{f}(x),$$

where $\sigma = \sqrt{2\alpha t}$.

So, to find u , all we need is to find a function whose Fourier transform is the product of two Fourier transforms!

In general, if $f, g : \mathbb{R} \rightarrow \mathbb{R}$,

then what is the function u s.t.

$$\hat{u} = (\mathcal{L}^{-1}) \cdot \hat{f} \cdot \hat{g} \quad ?$$

The answer is: the convolution $f * g$ of f and g .

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→ Def.: Let $f, g: \mathbb{R} \rightarrow \mathbb{R}$.

We define the convolution

$f * g$ of f and g as:

$f * g: \mathbb{R} \rightarrow \mathbb{R}$, with

$$f * g(x) = \int_{-\infty}^{+\infty} f(x-y) g(y) dy, \quad \forall x \in \mathbb{R}.$$

→ Basic property of convolution:

Let $f, g \in L^1(\mathbb{R})$. Then:

↑
so that we
can talk about
their Fourier transform.

$$\widehat{f * g} = 2\pi \cdot \widehat{f} \cdot \widehat{g}$$

(i.e., $\widehat{f * g}(x) = 2\pi \cdot \widehat{f}(x) \cdot \widehat{g}(x), \quad \forall x \in \mathbb{R}.$)

Proof: $\widehat{f * g}(x) = \frac{1}{2\pi} \cdot \int_{-\infty}^{+\infty} e^{-ixy} f * g(y) dy =$

$$= \frac{1}{2\pi} \int_{y=-\infty}^{+\infty} e^{-ixy} \cdot \left(\int_{u=-\infty}^{+\infty} f(y-u) g(u) du \right) dy =$$

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$$= \frac{1}{2\pi} \cdot \int_{u=-\infty}^{+\infty} g(u) \cdot \left(\int_{y=-\infty}^{+\infty} e^{-ix(y)} f(y-u) dy \right) du =$$

$$= \int_{u=-\infty}^{+\infty} g(u) \cdot e^{-ixu} \cdot \left(\frac{1}{2\pi} \cdot \int_{y=-\infty}^{+\infty} e^{-ixt} f(t) dt \right) du =$$

$\uparrow \uparrow$
 $f(x)$

$$= \hat{f}(x) \cdot \int_{u=-\infty}^{+\infty} e^{-ixu} g(u) du = 2\pi \cdot \hat{f}(x) \cdot \frac{1}{2\pi} \cdot \int_{-\infty}^{+\infty} e^{-ixu} g(u) du =$$

$\uparrow \uparrow$
 $\hat{g}(x)$

$$= 2\pi \cdot \hat{f}(x) \cdot \hat{g}(x), \quad x \in \mathbb{R}.$$



How to think about this: For every single x , the convolution $\hat{f} * \hat{g}(x)$ contains information from both f and g , blended together. The fourier transform decouples the two blended functions!

→ What does all this mean for the heat equation?

$$\text{We know that } \hat{u}(x,t) = 2\pi \cdot \hat{g}_0(x) \cdot \hat{f}(x)$$

by basic property

of convolution

(where $\sigma = \sqrt{2at}$).

$$\text{So, } u(x,t) = \frac{\hat{g}_0}{\sqrt{2at}} * f(x), \quad t \in \mathbb{R}, \quad x \in \mathbb{R}!$$

$$\hat{g}_0 * f(x)$$

the Gaussian with
standard deviation
 $\sqrt{2at}$

Even though we haven't seen the full advantage of this expression of u yet, we see that already it is better than the inversion formula; at least it makes it clear that the

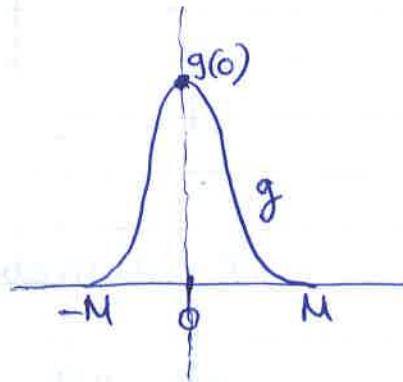
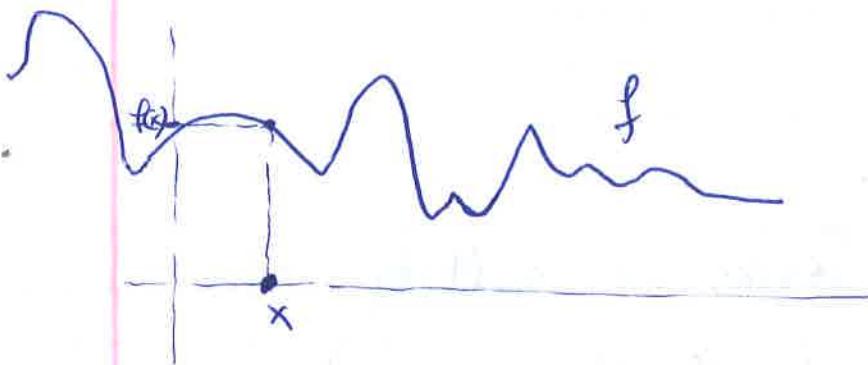
temperature u is real

(as there are no complex numbers involved in this expression).

→ What does $f * g(x)$ really mean?

$f * g(x)$ is really the average of f "around" x , against weight g (or, really, $g(-\cdot)$, g "flipped").

It may be easier to understand this when g is just a bump around 0:



$$f * g(x) = \int_{-\infty}^{+\infty} f(x-y) g(y) dy =$$

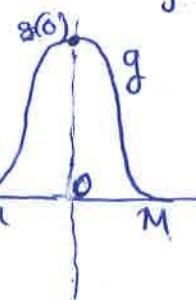
$\downarrow g=0 \text{ outside } [-N, N]$

$$= \int_{-N}^N f(x-y) g(y) dy$$

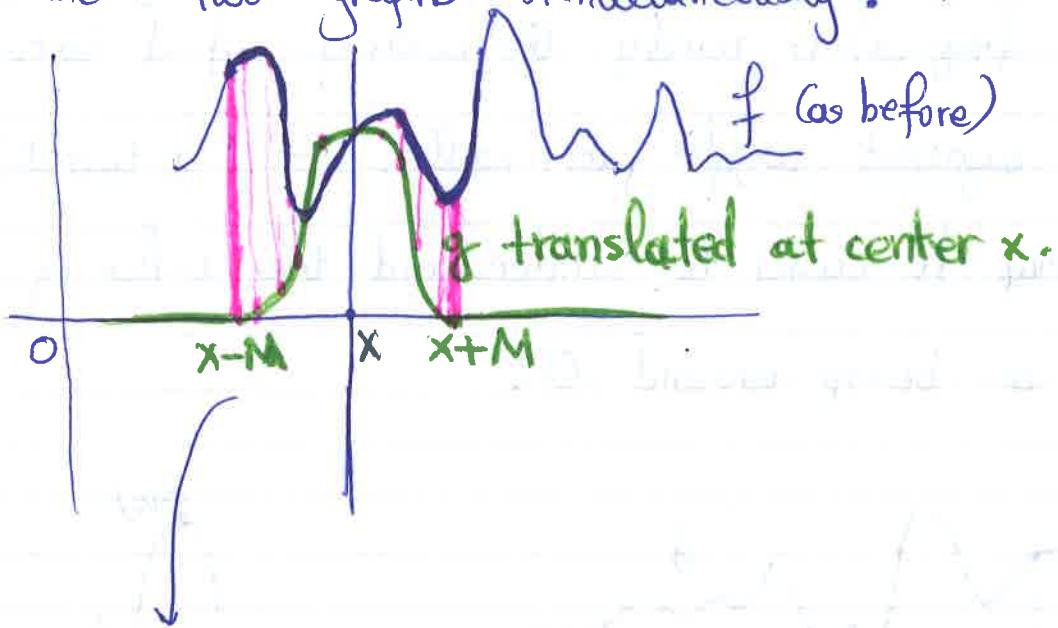
No matter what x is, $f(x)$ is multiplied with $g(0)$, $f(x+\varepsilon)$ is multiplied with $g(-\varepsilon)$, $f(x-\varepsilon)$ is multiplied with $g(\varepsilon) = g(-\varepsilon)$, etc...

$f(x-\varepsilon)$ is multiplied with $g(\varepsilon) = g(-\varepsilon)$, etc...

In this case where g is even, we translate also true for Gaussians!



to have centre x , and we look
at the two graphs simultaneously:



to calculate $f * g(x)$, we multiply
each value of f with the corresponding
value of this translated bump, and
we integrate. So, we pick up the
"average value" of f in $[x-M, x+M]$,

- modulo the fact that g is not 1 ^{translated}
on $[x-M, x+M]$, but rather a bump function,
a weight.
- modulo the fact that we are not

dividing with the length $2M$ of $[x-M, x+M]$

(if M was tiny, and g went up to height 1
only, then $f * g(x)$ would be tiny).

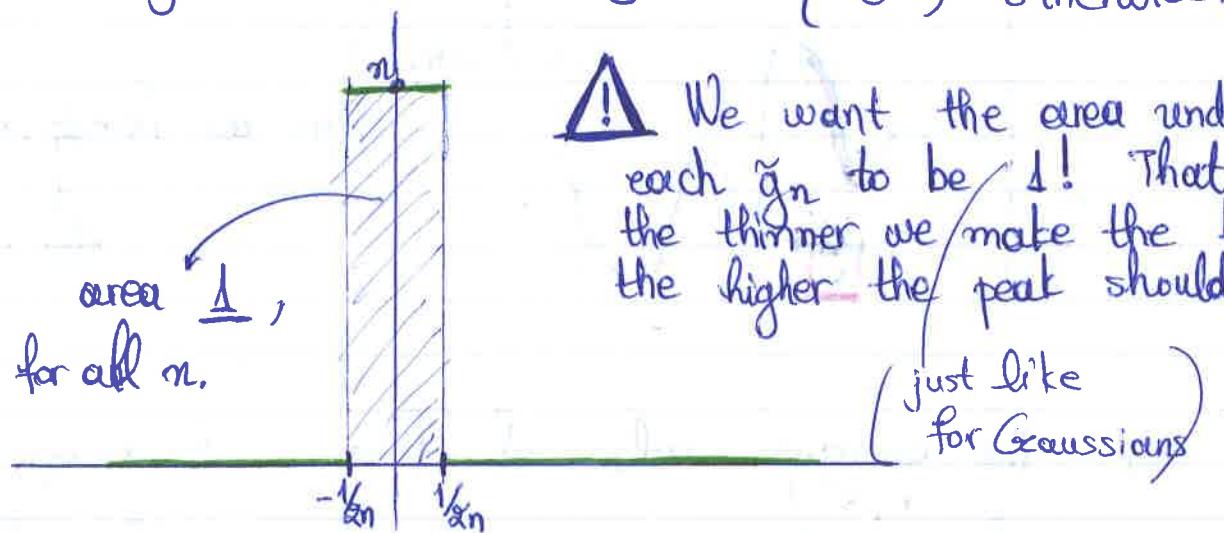
If g is not even, we first flip it around 0,

then we drag it from 0 to x ,

and that is the new weight against which we average f , to find $f*g(x)$.

→ Important example: (these functions approximate Gaussians!) $\frac{1}{n}$

for all $n \in \mathbb{N}$,
let $\tilde{g}_n : \mathbb{R} \rightarrow \mathbb{R}$, with $\tilde{g}_n(x) = \begin{cases} n, & x \in [-\frac{1}{2n}, \frac{1}{2n}] \\ 0, & \text{otherwise.} \end{cases}$



fix an $f : \mathbb{R} \rightarrow \mathbb{R}$. Let's calculate $f * \tilde{g}_n$:

$$f * \tilde{g}_n(x) = \int_{-\infty}^{+\infty} f(x-y) \tilde{g}_n(y) dy = \int_{-\frac{1}{2n}}^{\frac{1}{2n}} f(x-y) \cdot n dy =$$

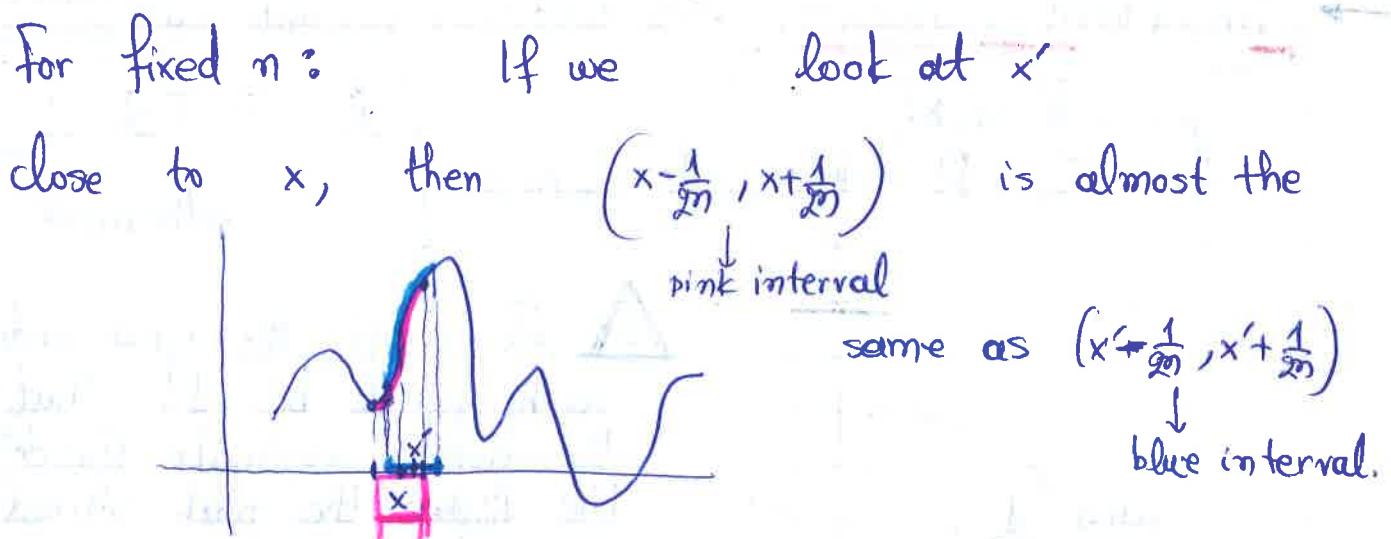
$$= n \cdot \int_{-\frac{1}{2n}}^{\frac{1}{2n}} f(x-y) dy = n \cdot \int_{x-\frac{1}{2n}}^{x+\frac{1}{2n}} f(u) du =$$

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$$= \frac{1}{\left[\text{length of } (x - \frac{1}{2n}, x + \frac{1}{2n}) \right]} \cdot \int_{x - \frac{1}{2n}}^{x + \frac{1}{2n}} f(u) du =$$

= the average value of f in $(x - \frac{1}{2n}, x + \frac{1}{2n})$.

Notice :



So, the average value of f on $(x - \frac{1}{n}, x + \frac{1}{n})$ is almost the same as the average value of f on $(x' - \frac{1}{2n}, x' + \frac{1}{2n})$, and, the larger the intervals we average in, the slower the change.

i.e. $f * \tilde{g}_n(x) \underset{\text{and, the larger the intervals we average in, the slower the change.}}{\approx} f * \tilde{g}_n(x')$!

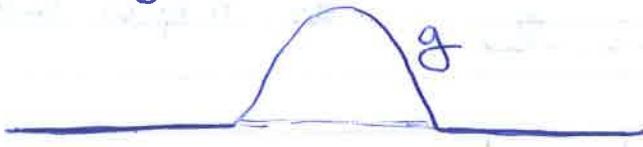
This means that the convolution of f with \tilde{g}_n gives a function that changes more slowly and smoothly than f .

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We expect that $f * g_n$ should be behaving more nicely than f for that reason... In general, the convolution of two functions inherits the best properties of both functions:

Fact (if you are interested):

Let g be a bump function with bounded support:



Let $f \in L^1(\mathbb{R})$
(potentially ugly).

- Then: • g continuous $\rightarrow f * g$ continuous.
- g smooth $\rightarrow f * g$ smooth.

A This means that, if we have an ugly f , and we want to make it smooth, we can just convolve it with a smooth g (with bounded support). We will soon see that we can pick so that $f * g(x)$ is pretty much the same as $f(x)$, $\forall x \in \mathbb{R}$. (will be Gaussians!)

(2)

We have explained that $f * g_n(x)$ is the average value of f in a small interval of length $\frac{1}{n}$ around x .

This is really how a picture can be blurred:

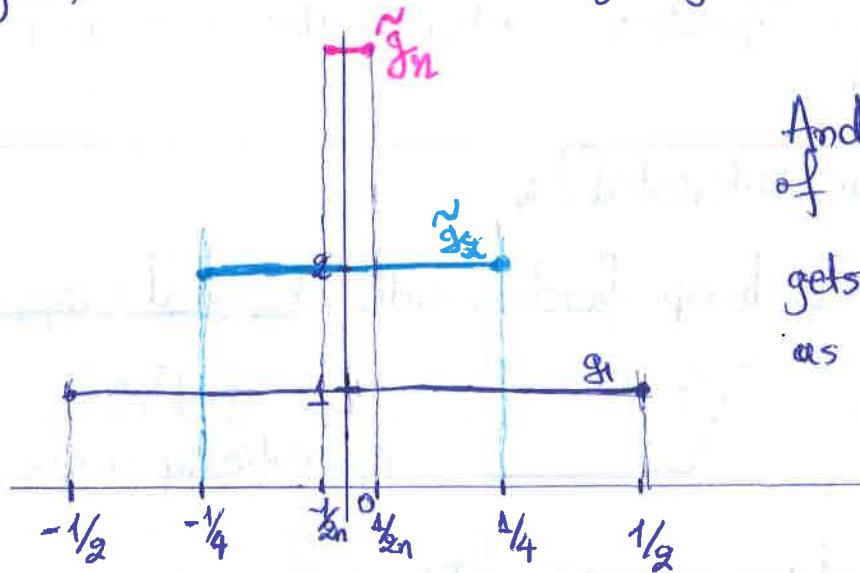
We "average" the picture around every point,
i.e. we convolve it with some \tilde{g}_n :

Case of

Large n : interesting in image processing.

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- ③ The larger n gets, the thinner the domain of \tilde{g}_n gets, and the taller \tilde{g}_n gets, too:



And the average of f on $(x-\frac{1}{n}, x+\frac{1}{n})$ tends to $f(x)$ as n gets larger.

So, when we convolve with \tilde{g}_n , the picture we get is very rough, very blurry, changes very slowly; the larger the n is, the more $f * \tilde{g}_n$ looks like f , i.e. the more precise the picture is!

This is all based on the fact that

$$f * \tilde{g}_n(x) \xrightarrow{n \rightarrow \infty} f(x), \forall x \in \mathbb{R}.$$

This would also be true if we considered smoothed-out versions of the \tilde{g}_n 's:

fact: Let $g_\sigma : \mathbb{R} \rightarrow \mathbb{R}$ be the Gaussian with standard deviation σ :

$$g_\sigma(x) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{x^2}{2\sigma^2}}$$

important here that $\int g_\sigma = 1$ & σ (just like for the \tilde{g}_n 's earlier)

Let $f \in L^1$.

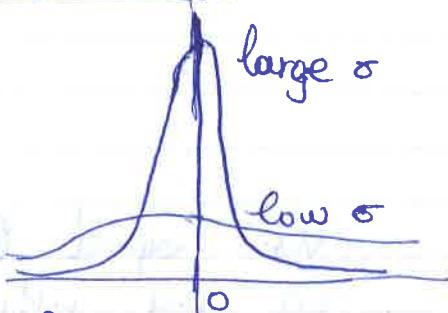
Then, $f * g_\sigma(x) \xrightarrow{\sigma \rightarrow 0} f(x)$, $\forall x \in \mathbb{R}$

(Even better, if you are interested: $\sup_{x \in \mathbb{R}} |f * g_\sigma(x) - f(x)| \xrightarrow{\sigma \rightarrow 0} 0$, i.e. we can find an $f * g_\sigma$ with a graph as close as we want to the graph of f .)

I.e.: As the standard deviation g_σ goes to 0,

g_σ becomes more and more localised, with a higher and higher peak

(looks more and more like the \tilde{g}_n 's from before, for large n 's),



and the "blurry" version of f , $f * g_\sigma$,

becomes more and more accurate: it tends to f .

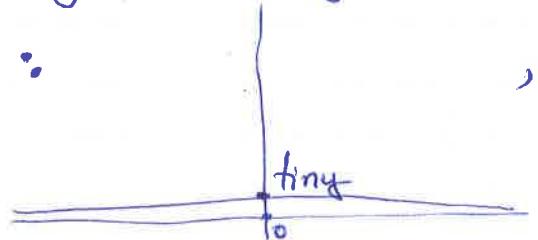


If you are interested in further reading:
The above happens because the g_σ 's form an "approximate identity".

Case of tiny σ (really: huge standard deviation):
interesting in heat diffusion.

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- ④ On the other hand, as σ gets larger and larger, the Gaussian g_0 looks like this:

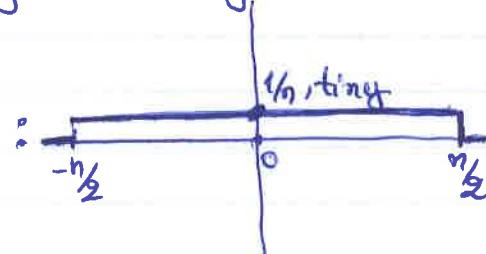


and $f * g_0$ is an incredibly blurred-out version of f : i.e., when σ is huge,

$f * g_0$ is almost constant.

We can see this by approximating such g_0 with

$$\tilde{g}_{1/n} = \begin{cases} \frac{1}{n}, & x \in [-\frac{n}{2}, \frac{n}{2}] \\ 0, & \text{otherwise} \end{cases}$$



for some huge n . Then:

$$|f * \tilde{g}_{1/n}(x)| = \left| \text{average of } f \text{ on } [x - \frac{n}{2}, x + \frac{n}{2}] \right| =$$

huge interval

$$= \frac{1}{n} \cdot \left| \int_{x - \frac{n}{2}}^{x + \frac{n}{2}} f(y) dy \right| \leq \frac{1}{n} \cdot \underbrace{\left(\int_{\mathbb{R}} |f| \right)}_{\text{fixed}} ,$$

very
small
when n
is huge.

pretty much
averaging f
over the whole of
 \mathbb{R} ! No matter what
 x is!

" " "
 $\int_{\mathbb{R}} f$, for many
 $x \in \mathbb{R}$, as n
is huge,

so $f * \tilde{g}_{1/n}(x)$ almost the
same for all those $x \in \mathbb{R}$

extra
information ...

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→ Another basic property of convolution:

$$\int f * g = \int f \cdot \int g, \quad \forall f, g \in L^1(\mathbb{R}).$$

Proof:

$$\begin{aligned} \int_{x=-\infty}^{+\infty} f * g(x) dx &= \int_{x=-\infty}^{+\infty} \left(\int_{y=-\infty}^{+\infty} f(x-y) g(y) dy \right) dx = \\ &= \int_{y=-\infty}^{+\infty} g(y) \left(\int_{x=-\infty}^{+\infty} f(x-y) dx \right) dy = \\ &\quad \text{Change order of integration} \\ &= \left(\int_{u=-\infty}^{+\infty} f(u) du \right) \cdot \left(\int_{y=-\infty}^{+\infty} g(y) dy \right) = \int f \cdot \int g \end{aligned}$$

→ Heat equation, part II:

We have so far seen that the solution $u(\cdot, t)$ to the heat equation on \mathbb{R}

$$a \Delta_x u(x, t) = \frac{\partial}{\partial t} u(x, t)$$

$$u(x, 0) = f(x) \quad \forall x \in \mathbb{R}, \quad \text{for } f \in L^1(\mathbb{R})$$

$$\text{is } u(\cdot, t) = f * g_{\sqrt{2at}}$$

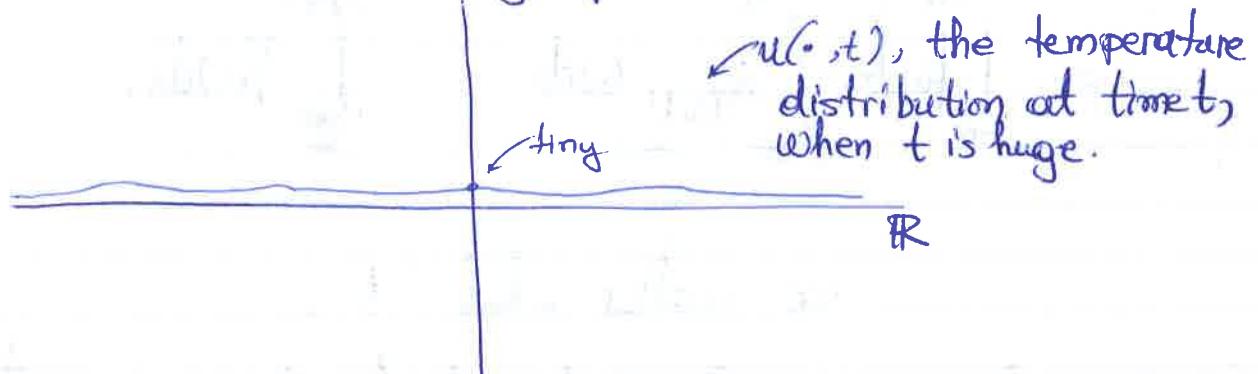
the Gaussian with standard deviation \sqrt{at} .

We will now explain mathematically the following 3 things that our intuition tells us:

- ① As time passes, the temperature should tend to be more and more uniformly distributed along our infinite "string".
- ② As time passes, there is no loss of heat i.e. the "mass" of temperature should be unchanging through time, i.e.

$$\int_{-\infty}^{\infty} u(x, t) dx$$
 should equal $\int_{-\infty}^{\infty} f(x) dx$
 at all times t .
- ③ By ① and ②, since the whole temperature "mass" should stay unchanged through time, yet with the temperature getting more and more uniformly distributed, we should be having that the temperature at time t gets closer and

closer to 0 as time t increases. I.e., we should have the following picture:



Explanations :

→ Of ①:

$$u(\cdot, t) = f * g_{\sqrt{2at}}$$

the temperature distribution at time t .

So, the larger t gets, the larger standard deviation the Gaussian $g_{\sqrt{2at}}$ has.

$u(\cdot, t)$

So, for t large, $f * g_{\sqrt{2at}}$ is almost constant (we saw this by approximating such Gaussians by $\tilde{g}_{1/n}$'s, n huge)

→ of ②: $\int_{-\infty}^{+\infty} u(x,t) dx = \int_{-\infty}^{+\infty} f * g_{\sqrt{2at}}(x) dx$ basic property

$$= \int_{-\infty}^{+\infty} f(x) dx \cdot \underbrace{\int_{-\infty}^{+\infty} g_{\sqrt{2at}}(x) dx}_{\text{no matter what } t \text{ is.}} = \int_{-\infty}^{+\infty} f(x) dx,$$

This of heat says that, indeed, there is no loss (or gain!)

→ of ③: We want to show that,

when t is huge, $\underbrace{u(x,t)}$ is small, for all x .
temperature of x at this fixed time t

Indeed: $(0 \leq) |u(x,t)| = |f * g_{\sqrt{2at}}(x)| =$

$$= \left| \int_{-\infty}^{+\infty} f(x-y) \cdot g_{\sqrt{2at}}(y) dy \right| =$$

$$= \left| \int_{-\infty}^{+\infty} f(x-y) \cdot \frac{1}{\sqrt{2\pi \sqrt{2at}^2}} \cdot e^{-\frac{y^2}{2\sqrt{2at}^2}} dy \right| \leftarrow$$

$$\leq \frac{1}{\sqrt{2\pi \cdot 2at}} \cdot \int_{-\infty}^{+\infty} |f(x-y)| \cdot \left| e^{-\frac{y^2}{4at}} \right| dy =$$

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$$= \frac{1}{\sqrt{4\pi n \cdot t}} \cdot \int_{-\infty}^{+\infty} |f(x-y)| \cdot \left| e^{-\frac{y^2}{4at}} \right| dy \leq$$

$$\leq \frac{1}{\sqrt{4\pi n \cdot t}} \cdot \int_{-\infty}^{+\infty} |f(x-y)| dy = \frac{1}{\sqrt{4\pi n}} \cdot \frac{1}{\sqrt{t}} \cdot \underbrace{\int |f|}_{<+\infty, \text{ as } f \in L^1(\mathbb{R})} =$$

$$= (\text{constant}) \cdot \frac{1}{\sqrt{t}}$$

i.e., $0 \leq |u(x,t)| \leq (\text{constant}) \cdot \frac{1}{\sqrt{t}}$ at time t ,

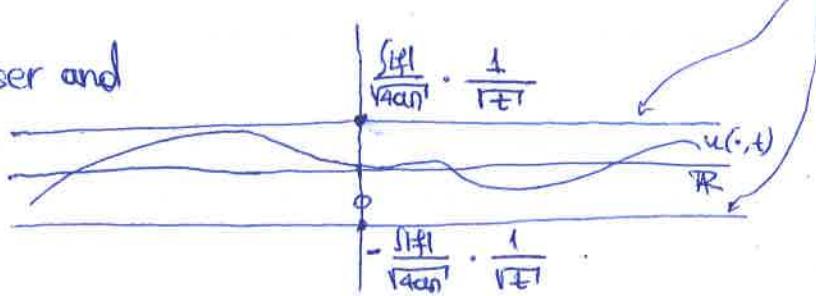
$\underbrace{|u(x,t)|}_{\text{Temperature at } x} \leq \underbrace{\frac{1}{\sqrt{t}} \int |f|}_{\frac{1}{\sqrt{4\pi n}} \int |f|}$

no matter what x is.

So, the temperature gets uniformly lower
(and tends to 0) as time evolves.

Indeed, for time t , $u(\cdot, t)$ lives between these two horizontal lines:

These lines get closer and closer to the x -axis as time increases, so $u(\cdot, t)$ is squashed down to 0.



→ Conclusions:

We start with a temperature distribution $f: \mathbb{R} \rightarrow \mathbb{R}$.

for $t > 0$ but small, the temperature distribution is $f * g_{\sqrt{2at}}$, where the standard deviation of the Gaussian $g_{\sqrt{2at}}$ is $\sqrt{2at}$, small. So, the temperature distribution is a very slightly blurred f (i.e., almost the same as the original f).

As time evolves, t gets larger, so $f * g_{\sqrt{2at}}$

takes values that are really averages of f on larger and larger intervals. So, $f * g_{\sqrt{2at}}$ tends to be uniformly distributed, and

$$\text{in fact } [f] = \frac{\int f}{\text{length}} = \frac{(\text{finite number})}{\text{too}} = 0.$$

the largest possible interval.